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Concerning the Linear Dependence of Integer Translates of Exponential Box Splines

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TO MY ERSTWHILE TEACHER, PROFESSOR K. M. DAS, IN MEMORIAM

Let $B_{\Xi,\lambda}$ be the exponential box spline associated with $\lambda \in \mathbf{C}^n$ and an $s \times n$ rational matrix with rank s and non-zero columns. Sufficient conditions are provided for the kernel space

$$K(B_{\Xi,\lambda}) := \left\{ a : \mathbf{Z}^s \to \mathbf{C} : \sum_{j \in \mathbf{Z}^s} a(j)\, B_{\Xi,\lambda}(\cdot - j) = 0 \right\}$$

to be (i) trivial and (ii) finite dimensional. While these results extend the corresponding theorems known for integer matrices, the methods of proof are discernibly different.

1. INTRODUCTION

Let ψ be a compactly supported distribution on $\mathbf{R}^s$ and $\hat{\psi}$ its Fourier transform. We define the sequence space

$$K(\psi) := \left\{ a : \mathbf{Z}^s \to \mathbf{C} : \sum_{j \in \mathbf{Z}^s} a(j)\, \psi(\cdot - j) = 0 \right\}$$

and the set

$$N(\psi) := \{ \theta \in \mathbf{C}^s : \hat{\psi}(\theta + 2\pi j) = 0 \qquad \text{for all} \quad j \in \mathbf{Z}^s \}.$$

When $K(\psi)$ is the trivial space $\{0\}$, we say that the *integer translates of* ψ *are linearly independent*. This phenomenon of linear independence of translates has received a good deal of attention in the literature. For instance,

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it is known [12, 4, 15] that $K(\psi)$ is trivial if and only if it contains no exponential sequences; in turn, this is equivalent to the set $N(\psi)$ being empty [15]. Apart from their obvious generality, such results also shed considerable light when one studies particular distributions about whose structure and properties more precise information is available.

An especially interesting choice of the distribution ψ is provided by the *exponential box spline* whose study was initiated in [14] and carried out in greater detail in [1, 6]. To recall its definition, let Ξ be an $s \times n$ real matrix with non-zero columns and let $\lambda \in \mathbf{C}^n$. The exponential box spline $B_{\Xi,\lambda}$ associated with Ξ and λ is the compactly supported distribution on $\mathbf{R}^s$ given by the rule

$$\langle B_{\Xi,\lambda}, f \rangle := \int_{[0,1]^n} e^{\lambda \cdot t} f(\Xi t)\, dt, \qquad f \in C(\mathbf{R}^s). \tag{1.1}$$

An important special case, namely $\lambda = 0$, corresponds to the well-known *polynomial box spline* B_Ξ introduced earlier in [2, 3]. Box splines possess a remarkably rich structure and the study of their various properties constitutes a dominant theme in the theory of multivariate splines. Although Definition 1.1 imposes no restrictions on the nature of the matrix Ξ, most of the hitherto available results on box splines presuppose that Ξ is an integer matrix (i.e., all entries in Ξ are integers). In particular, the question of linear independence of such polynomial box spline translates is addressed in [3, 4, 10] and that of their exponential counterparts is examined in [6, 14, 16].

Yet another aspect of the sequence space $K(\psi)$ that has also merited attention is its dimensionality to which we now turn. The most general result in this connection is the following; the necessity part of this theorem was proved in [15] whereas the sufficiency part was obtained in [7].

THEOREM 1.1. *Let ψ be a compactly supported distribution on $\mathbf{R}^s$. Then $K(\psi)$ is finite dimensional if and only if $N(\psi)/2\pi\mathbf{Z}^s$ is a finite set.*

Applications of Theorem 1.1 to box splines associated with integer matrices are also discussed in [7] as well as in [8, 9]. These results are stated elsewhere in this article.

The present paper has a twofold objective. First, we discuss the linear independence of the integer translates of $B_{\Xi,\lambda}$ when Ξ is a rational matrix. The main theorem in this regard is a sufficient condition for $K(B_{\Xi,\lambda})$ to be trivial. The corresponding problem for polynomial box splines was treated in [11] to which work the first part of this paper may be considered a sequel. The second part of the paper is motivated by [7–9] and is devoted to the derivation of a sufficient condition for the space $K(B_{\Xi,\lambda})$ to have finite dimension. Here again, Ξ is assumed to be a rational matrix. In either case, effecting the transition between integer and rational matrices calls for the pursuance of a different tack. We are thereby led to suitable generalizations of the corresponding results for integer matrices as well as new proofs for these results.

In closing this section, the author thanks Professors W. Dahmen, R. Q. Jia, and C. A. Micchelli for providing him with preprints of [7, 8] and a copy of [9].

2. A Sufficient Condition for Linear Independence

Let $B_{\Xi,\lambda}$ be the exponential box spline associated with $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbf{C}^n$ and an $s \times n$ real matrix Ξ with non-zero columns $x_1, x_2, ..., x_n$. The Fourier transform of $B_{\Xi,\lambda}$ is given by (see [14])

$$\hat{B}_{\Xi,\lambda}(x) := \prod_{k=1}^{n} \frac{e^{\lambda_k - ix_k \cdot x} - 1}{\lambda_k - ix_k \cdot x}, \tag{2.1}$$

and if rank $\Xi = s$, then $B_{\Xi,\lambda}$ is a piecewise exponential polynomial function on $\mathbf{R}^s$.

We first note that as a consequence of (2.1), $\theta \in N(B_{\Xi,\lambda})$ if and only if for each $l \in \mathbf{Z}^s$, there exists a j, $1 \leqslant j \leqslant n$, such that

$$i\frac{\lambda_j}{2\pi} + \left(\frac{\theta}{2\pi} + l\right) \cdot x_j \in \mathbf{Z} \backslash \{0\}. \tag{2.2}$$

In particular, if $\lambda = 0$, then

$$N(B_{\Xi,\lambda}) \subseteq \mathbf{R}^s. \tag{2.3}$$

Given the matrix Ξ, we define

$$\mathscr{B}(\Xi) := \{Y: Y \text{ is an invertible } s \times s \text{ submatrix of } \Xi\},$$

and for $\phi \in \mathbf{C}^s$,

$$\Gamma_\phi(\Xi) := \{x_j: \phi \cdot x_j = \lambda_j\}.$$

Further, let

$$\Theta_\lambda(\Xi) := \{\phi \in \mathbf{C}^s: \operatorname{span} \Gamma_\phi(\Xi) = \mathbf{R}^s\}.$$

If Ξ is an integer matrix, then the linear independence of the integer translates of $B_{\Xi,\lambda}$ is characterized by the following theorem [16, Theorem 4.3; 6, Corollary 4.1].

THEOREM 2.1. *Let Ξ be an $s \times n$ integer matrix with rank s and non-zero columns and let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbf{C}^n$. Then the integer translates of $B_{\Xi,\lambda}$ are linearly independent if and only if the following hold*:

(a) $|\det Y| = 1$ *for all* $Y \in \mathscr{B}(\Xi)$;

(b) $\hat{B}_{\Xi,\lambda}(-i\phi) \neq 0$ *for all* $\phi \in \Theta_\lambda(\Xi)$.

Remark 2.2. By (2.1), $\hat{B}_{\Xi,\lambda}(-i\phi) \neq 0$ if and only if

$$i\frac{\lambda_k}{2\pi} - i\frac{\phi}{2\pi} \cdot x_k \notin \mathbf{Z}\backslash\{0\}, \qquad k = 1, 2, ..., n. \tag{2.4}$$

Remark 2.3. (i) If $\lambda \in \mathbf{R}^n$, then $\Theta_\lambda(\Xi) \subseteq \mathbf{R}^s$; so $\hat{B}_{\Xi,\lambda}(-i\phi) \neq 0$ for $\phi \in \Theta_\lambda(\Xi)$.

(ii) If $n = s = \operatorname{rank} \Xi$, then $\Theta_\lambda(\Xi) = \{\check{\phi}\}$ is a singleton set and $\hat{B}_{\Xi,\lambda}(-i\check{\phi}) = 1$.

As indicated in the previous section, a necessary and sufficient condition for the linear independence of the integer translates of $B_{\Xi,\lambda}$ is that the set $N(B_{\Xi,\lambda})$ be empty. In view of (2.2), this is tantamount to requiring for each $\theta \in \mathbf{C}^s$, an $l \in \mathbf{Z}^s$ such that

$$i\frac{\lambda_k}{2\pi} + \left(\frac{\theta}{2\pi} + l\right) \cdot x_k \notin \mathbf{Z}\backslash\{0\} \qquad \text{for all} \quad k = 1, 2, ..., n. \tag{2.5}$$

Turning to our main results of this section, let us begin with the univariate case ($s = 1$).

THEOREM 2.4. *Let Ξ be a $1 \times n$ rational matrix with non-zero entries $x_1, x_2, ..., x_n$. Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbf{C}^n$ and suppose that*

(a) $x_k = 1/q_k$, $q_k \in \mathbf{Z}\backslash\{0\}$ *for all* k;

(b) $\hat{B}_{\Xi,\lambda}(-i\phi) \neq 0$ *for all* $\phi \in \Theta_\lambda(\Xi)$.

Then the integer translates of $B_{\Xi,\lambda}$ are linearly independent.

Proof. Let $\theta = 2\pi(\alpha + i\beta) \in \mathbf{C}$ be given and suppose that $\lambda_k = 2\pi(\sigma_k + it_k)$. By (2.5), it suffices to find an $l \in \mathbf{Z}$ such that

$$i\left(\sigma_k + \frac{\beta}{q_k}\right) + \frac{\alpha + l - q_k t_k}{q_k} \notin \mathbf{Z}\backslash\{0\}, \qquad k = 1, 2, ..., n. \tag{2.6}$$

Assume, without loss of generality,

$$\sigma_k + \frac{\beta}{q_k} = 0, \qquad k = 1, 2, ..., n. \tag{2.7}$$

Noting that

$$\Theta_\lambda(\Xi)=\left\{\frac{\lambda_j}{x_j}: 1\leqslant j\leqslant n\right\},$$

one can deduce from supposition (b), (2.4), and (2.7) that for each $j=1, 2, ..., n$,

$$\frac{q_j t_j - q_k t_k}{q_k}\notin \mathbf{Z}\backslash\{0\}, \qquad k=1, 2, ..., n. \tag{2.8}$$

Now, to prove (2.6), consider the following cases.

Case I. $\alpha - q_k t_k \notin \mathbf{Z}$ for all $k=1, 2, ..., n$.

In this case, (2.6) holds for $l=0$.

Case II. There is a j such that $\alpha - q_j t_j \in \mathbf{Z}$.

If so, choose $l=q_j t_j-\alpha$ and (2.6) follows from (2.8). ∎

Remark 2.5. An adaptation of the proof of [11, Theorem 2.1] shows that if Ξ is a $1\times n$ rational matrix with non-zero entries x_k and $\lambda\in\mathbf{R}^n$, then the integer translates of $B_{\Xi,\lambda}$ are linearly independent if and only if $x_k\notin\mathbf{Z}\backslash\{-1, 1\}$ for each $k=1, 2, ..., n$. However, the next example shows that if $\lambda\notin\mathbf{R}^n$, then the conditions $\hat{B}_{\Xi,\lambda}(-i\phi)\neq 0$ for $\phi\in\Theta_\lambda(\Xi)$ and $x_k\notin\mathbf{Z}\backslash\{-1, 1\}$ for all k do not ensure linear independence.

EXAMPLE 2.6. Let $x_1=x_2=3/2$, $\lambda_1=2\pi i$, $\lambda_2=3\pi i$ so that $\Theta_\lambda(\Xi)=\{4\pi i/3, 2\pi i\}$. It is easy to verify that the two conditions stated above are satisfied. However, if $\theta=8\pi/3$ and $l\in\mathbf{Z}$, then

$$i\frac{\lambda_1}{2\pi}+\left(\frac{\theta}{2\pi}+l\right)x_1=\frac{3l+2}{2} \tag{2.9}$$

and

$$i\frac{\lambda_2}{2\pi}+\left(\frac{\theta}{2\pi}+l\right)x_2=\frac{3l+1}{2}. \tag{2.10}$$

For any $l\in\mathbf{Z}$, one of either (2.9) or (2.10) is a non-zero integer. By (2.5), the integer translates of $B_{\Xi,\lambda}$ are linearly dependent.

The following example shows that even assuming condition (a), condition (b) of Theorem 2.4 is not necessary for linear independence. (Contrast this with the situation for integer matrices.)

EXAMPLE 2.7. Let $x_1 = \frac{1}{2}$, $x_2 = 1$, $\lambda_1 = 0$, $\lambda_2 = 2\pi i$, so $\Theta_\lambda(\Xi) = \{0, 2\pi i\}$. It is easy to check that $\hat{B}_{\Xi,\lambda}(0) = 0$. Yet, we claim that the integer translates of $B_{\Xi,\lambda}$ are linearly independent.

To that end, let $\theta = 2\pi(\alpha + i\beta) \in \mathbf{C}$, $\alpha, \beta \in \mathbf{R}$. If $\beta \neq 0$, or if α is not an integer, then (2.5) is satisfied for $l = 0$. On the other hand, if $\beta = 0$ and α is an integer, then (2.5) is satisfied for $l = 1 - \alpha$. Thus (2.5) is satisfied for all $\theta \in \mathbf{C}$, validating our claim.

Remark 2.8. Suppose that Ξ has rank s. If V is an invertible $s \times s$ real matrix and V^{T} its transpose, then we have the following:

(i) $\phi \in \Theta_\lambda(V\Xi)$ if and only if $V^{\mathrm{T}}\phi \in \Theta_\lambda(\Xi)$.

(ii) It follows from (1.1) and (2.1) that

$$B_{\Xi,\lambda}(x) = |\det V|\, B_{V\Xi,\lambda}(Vx) \tag{2.11}$$

and

$$\hat{B}_{\Xi,\lambda}(V^{\mathrm{T}}x) = \hat{B}_{V\Xi,\lambda}(x). \tag{2.12}$$

In particular, if V is a *unimodular* (i.e., $|\det V| = 1$) $s \times s$ *integer matrix*, then $V\mathbf{Z}^s = \mathbf{Z}^s$; so (2.11) ensures that the integer translates of $B_{\Xi,\lambda}$ are linearly independent if and only if the integer translates of $B_{V\Xi,\lambda}$ are.

THEOREM 2.9. *Let Ξ be an $s \times n$ rational matrix with rank s and non-zero columns $x_1, x_2, ..., x_n$. Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbf{C}^n$ and assume that*

(a) *Y^{-1} is an integer matrix for each $Y \in \mathscr{B}(\Xi)$;*

(b) *$\hat{B}_{\Xi,\lambda}(-i\phi) \neq 0$ for all $\phi \in \Theta_\lambda(\Xi)$.*

Then the integer translates of $B_{\Xi,\lambda}$ are linearly independent.

Proof. The proof proceeds by induction on s and n. Theorem 2.4 covers the case $s = 1$. Assume inductively that the result is true for $s - 1$.

Next, the case $n = s$ can be dealt with as follows. Given $\theta \in \mathbf{C}^s$, suppose that $\theta = 2\pi(\alpha + i\beta)$, $\alpha, \beta \in \mathbf{R}^s$, and $\lambda_k = 2\pi(\sigma_k + it_k)$, $k = 1, 2, ..., s$, with $\sigma_k, t_k \in \mathbf{R}$. Solve the system of equations for $l = (l_1, l_2, ..., l_s)$,

$$\sum_{\nu=1}^{s} l_\nu x_k^\nu = \lfloor t_k - \alpha \cdot x_k \rfloor, \qquad k = 1, 2, ..., s,$$

where $\lfloor a \rfloor$ denotes the integer part of a. Since Ξ^{-1} is now an integer matrix, $l \in \mathbf{Z}^s$ and (2.5) is satisfied for this l. This proves the result for $n = s$ so its validity may be assumed for $n - 1$.

Now, let Ξ be an $s \times n$ matrix. Without loss of generality, we may assume that there exists a $Y \in \mathscr{B}(\Xi)$ containing x_1 as one of its columns and also that the $s \times (n-1)$ matrix with columns $x_2, ..., x_n$ has rank s. Let $x_1 = (1/m)\, y_1$, where $y_1 \in \mathbf{Z}^s$ and $m \in \mathbf{Z}\backslash\{0\}$. As in [11, Theorem 2.4], we follow the proof of Theorem II.2 in [13] to obtain an $s \times s$ unimodular integer matrix V such that

$$Vy_1 = (\tilde{y}_1^1, 0, 0, ..., 0)^{\mathrm{T}}, \qquad \tilde{y}_1^1 \in \mathbf{Z}\backslash\{0\}.$$

Let $\tilde{\Xi} := V\Xi$ and denote its columns by $\tilde{x}_i$, $i = 1, ..., n$. Note that $\tilde{\Xi}$ is a rational matrix with non-zero columns and rank s; moreover, it has the form

$$\tilde{\Xi} = \begin{pmatrix} \tilde{x}_1^1 & \tilde{x}_2^1 & \cdots & \tilde{x}_n^1 \\ 0 & \tilde{x}_2^2 & \cdots & \tilde{x}_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \tilde{x}_2^s & \cdots & \tilde{x}_n^s \end{pmatrix}$$

where $\tilde{x}_1^1 = \tilde{y}_1^1/m$. Furthermore, the $s \times (n-1)$ matrix with columns $\tilde{x}_k$, $k = 2, ..., n$, has rank s. In view of Remark 2.8(ii), it suffices to prove the theorem for $B_{\tilde{\Xi},\lambda}$.

If $Z \in \mathscr{B}(\tilde{\Xi})$, then $Z = VW$ for some $W \in \mathscr{B}(\Xi)$; since W^{-1} and V^{-1} are integer matrices, so is Z^{-1}. This shows that $\tilde{\Xi}$ satisfies condition (a) of the theorem. By Remark 2.8(i) and (2.12), $B_{\tilde{\Xi},\lambda}$ satisfies assumption (b) as well. Therefore, the induction hypotheses can be (and are) applied to submatrices of $\tilde{\Xi}$.

Let $r_2, r_3, ..., r_s$ be integers such that $1 < r_2 < r_3 < \cdots < r_s \leqslant n$. Note that if the matrix

$$\begin{pmatrix} \tilde{x}_1^1 & \tilde{x}_{r_2}^1 & \cdots & \tilde{x}_{r_s}^1 \\ 0 & \tilde{x}_{r_2}^2 & \cdots & \tilde{x}_{r_s}^2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \tilde{x}_{r_2}^s & \cdots & \tilde{x}_{r_s}^s \end{pmatrix} \tag{2.13}$$

is invertible, then the inverse, by hypothesis, is an integer matrix. Therefore, $(\tilde{x}_1^1)^{-1} \in \mathbf{Z}$ and the inverse of the matrix

$$\begin{pmatrix} \tilde{x}_{r_2}^2 & \cdots & \tilde{x}_{r_s}^2 \\ \vdots & \ddots & \vdots \\ \tilde{x}_{r_2}^s & \cdots & \tilde{x}_{r_s}^s \end{pmatrix} \tag{2.14}$$

exists and is an integer matrix.

Now, let $\theta = (\theta_1, \theta_2, ..., \theta_s) \in \mathbf{C}^s$ be given. We show that (2.5) holds. There are two possible cases.

Case I.

$$\frac{\theta_1}{2\pi}+i\frac{\lambda_1}{2\pi\tilde{x}_1^1}\notin \mathbf{Z}. \tag{2.15}$$

Let $\tilde{\Xi}'$ be the $s\times(n-1)$ submatrix of $\tilde{\Xi}$ whose columns comprise $\tilde{x}_k$, $k=2, 3, ..., n$. Then $\tilde{\Xi}'$ satisfies condition (a). Since $B_{\tilde{\Xi},\lambda}$ satisfies (b) and $\Theta_\lambda(\tilde{\Xi}')\subseteq\Theta_\lambda(\tilde{\Xi})$, one infers from (2.1) that $B_{\tilde{\Xi}',\lambda}$ satisfies (b) as well. The induction hypothesis, applied to $B_{\tilde{\Xi}',\lambda}$, furnishes an $l=(l_1, l_2, ..., l_s)\in\mathbf{Z}^s$ such that

$$i\frac{\lambda_k}{2\pi}+\left(\frac{\theta}{2\pi}+l\right)\cdot\tilde{x}_k\notin\mathbf{Z}\backslash\{0\}, \qquad k=2, 3, ..., n. \tag{2.16}$$

Further,

$$i\frac{\lambda_1}{2\pi}+\left(\frac{\theta}{2\pi}+l\right)\cdot\tilde{x}_1=\left(i\frac{\lambda_1}{2\pi\tilde{x}_1^1}+\frac{\theta_1}{2\pi}+l_1\right)\tilde{x}_1^1\notin\mathbf{Z} \tag{2.17}$$

as l_1 and $(\tilde{x}_1^1)^{-1}$ are integers and (2.15) holds.

This proves (2.5) in this case.

Case II.

$$\frac{\theta_1}{2\pi}+i\frac{\lambda_1}{2\pi\tilde{x}_1^1}\in\mathbf{Z}.$$

Rearranging columns $2, 3, ..., n$ of $\tilde{\Xi}$ (*and* the corresponding λ_k's), if necessary, assume that for each $k=2, ..., m$, $\tilde{x}_k^j\neq 0$ for some $j=2, ..., s$ and for each $k=m+1, ..., n$, $\tilde{x}_k^j=0$ for all $j=2, ..., s$ and $\tilde{x}_k^1\neq 0$.

Consider the $(s-1)\times(m-1)$ submatrix $\tilde{\Xi}''$of $\tilde{\Xi}$ whose columns are $(\tilde{x}_k^2, ..., \tilde{x}_k^s)^{\mathrm{T}}$, $k=2, 3, ..., m$. Then $\tilde{\Xi}''$ has non-zero columns and rank $s-1$. If $Z\in\mathscr{B}(\tilde{\Xi}'')$, then it is of the form (2.14) and can be extended to an element in $\mathscr{B}(\tilde{\Xi})$ of the form (2.13). Consequently, Z^{-1} is an integer matrix. This shows that $\tilde{\Xi}''$ satisfies condition (a). Let $\mu:=(\mu_2, ..., \mu_m)\in\mathbf{C}^{m-1}$, where

$$\mu_k:=\lambda_k-\frac{\lambda_1}{\tilde{x}_1^1}\tilde{x}_k^1, \qquad k=2, 3, ..., m. \tag{2.18}$$

If

$$\tilde{\phi}''=(\tilde{\phi}_2'', ..., \tilde{\phi}_s'')\in\Theta_\mu(\tilde{\Xi}''),$$

then there exists an $(s-1)\times(s-1)$ matrix in $\mathscr{B}(\tilde{\Xi}'')$ with columns $(\tilde{x}^2_{r_j}, ..., \tilde{x}^s_{r_j})^{\mathrm{T}}$, $j=2, 3, ..., s$, such that

$$\sum_{v=2}^{s} \tilde{\phi}''_v \tilde{x}^v_{r_j} = \mu_{r_j}, \qquad j=2, 3, ..., s. \tag{2.19}$$

Define $\tilde{\phi} := (\lambda_1/\tilde{x}^1_1, \tilde{\phi}''_2, ..., \tilde{\phi}''_s) \in \mathbf{C}^s$, so

$$\tilde{\phi} \cdot \tilde{x}_1 = \lambda_1. \tag{2.20}$$

Since the $s\times s$ matrix with columns $(\tilde{x}_{r_j})_{j=1}^{s}$ $(r_1 := 1)$ belongs to $\mathscr{B}(\tilde{\Xi})$, (2.18), (2.19), and (2.20) imply that $\tilde{\phi} \in \Theta_\lambda(\tilde{\Xi})$ and so $\hat{B}_{\tilde{\Xi},\lambda}(-i\tilde{\phi}) \neq 0$. Therefore, from (2.4) and (2.18), $\hat{B}_{\tilde{\Xi}'',\mu}(-i\tilde{\phi}'') \neq 0$, showing that $B_{\tilde{\Xi},\mu}$ satisfies condition (b). By the induction hypothesis applied to $B_{\tilde{\Xi}'',\mu}$, one can find an $(l_2, ..., l_s) \in \mathbf{Z}^{s-1}$ such that

$$i\frac{\mu_k}{2\pi} + \sum_{v=2}^{s}\left(\frac{\theta_v}{2\pi} + l_v\right)\tilde{x}^v_k \notin \mathbf{Z}\backslash\{0\}, \qquad k=2, 3, ..., m. \tag{2.21}$$

Let $l_1 = -(\theta_1/2\pi) - i(\lambda_1/2\pi\tilde{x}^1_1)$ so that $l := (l_1, l_2, ..., l_s) \in \mathbf{Z}^s$. Clearly,

$$i\frac{\lambda_1}{2\pi} + \left(\frac{\theta}{2\pi} + l\right)\cdot\tilde{x}_1 = 0, \tag{2.22}$$

and from (2.18) and (2.21),

$$i\frac{\lambda_k}{2\pi} + \left(\frac{\theta}{2\pi} + l\right)\cdot\tilde{x}_k \notin \mathbf{Z}\backslash\{0\}, \qquad k=2, 3, ..., m. \tag{2.23}$$

Now, for $k=m+1, ..., n$, note that

$$\tilde{x}_k = \left(\frac{\tilde{x}^1_k}{\tilde{x}^1_1}\right)\tilde{x}_1 \tag{2.24}$$

and

$$i\frac{\lambda_k}{2\pi} + \left(\frac{\theta}{2\pi} + l\right)\cdot\tilde{x}_k = i\frac{\lambda_k}{2\pi} - i\frac{\lambda_1}{2\pi\tilde{x}^1_1}\tilde{x}^1_k. \tag{2.25}$$

Choose $\rho \in \Theta_\lambda(\tilde{\Xi})$ so that

$$\rho\cdot\tilde{x}_1 = \lambda_1. \tag{2.26}$$

As $\hat{B}_{\tilde{\Xi},\lambda}(-i\rho) \neq 0$, it follows from (2.4), (2.24), and (2.26) that

$$i\frac{\lambda_k}{2\pi} - i\frac{\lambda_1}{2\pi\tilde{x}^1_1}\tilde{x}^1_k \notin \mathbf{Z}\backslash\{0\}, \qquad k=m+1, ..., n. \tag{2.27}$$

From (2.27), (2.25), (2.23), and (2.22), we conclude that (2.5) holds once again and this completes the proof.

Remark 2.10. In comparing Theorem 2.9 with Theorem 2.1, note that if Ξ is an integer matrix and $Y \in \mathscr{B}(\Xi)$, then Y^{-1} is an integer matrix if and only if $|\det Y| = 1$.

Remark 2.11. In general, condition (a) of Theorem 2.9 is not necessary for linear independence. As a simple example, consider

$$\Xi = \begin{pmatrix} 3/2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \lambda = 0.$$

Obviously, Ξ^{-1} is not an integer matrix but for any $\theta = (\theta_1, \theta_2) \in \mathbf{R}^2$, $0 \leqslant \theta_i < 2\pi$, there is an $(l_1, 0) \in \mathbf{Z}^2$ such that (2.5) holds. This shows (by (2.3)) that $N(B_\Xi)/2\pi\mathbf{Z}^2$ and hence $N(B_\Xi)$ is an empty set. That is, the integer translates of B_Ξ are linearly independent. (For more discussion in this regard, see [11, Section 3].)

Remark 2.12. If $\lambda \in \mathbf{R}^n$, then Remark 2.3(i) and Theorem 2.9 show that the integer translates of $B_{\Xi,\lambda}$ are linearly independent provided Y^{-1} is an integer matrix for every $Y \in \mathscr{B}(\Xi)$. In particular, if $\lambda = 0$, then one recovers [11, Theorem 2.4].

Remark 2.13. In view of what is to follow in Section 4, we wish to reformulate condition (b) given in Theorems 2.1 and 2.9. Let $B_{\Xi,\lambda}$ be the exponential box spline associated with an $s \times n$ matrix Ξ and $\lambda \in \mathbf{C}^n$. Suppose that Y is an $s \times k$ submatrix of Ξ. We denote by λ_Y the vector in $\mathbf{C}^k$ whose components are those λ_j's corresponding to the columns of Y. We employ the notation $y \in Y$ to denote that y is a column of Y and use $(\Xi \backslash Y)$ to denote the $s \times (n-k)$ submatrix of Ξ whose columns consist of those $\xi \in \Xi$ which are not columns of Y. It must be noted that Y and $(\Xi \backslash Y)$ *can* have columns in common. Further, we let

$$\langle Y \rangle := \left\{ \sum a_y y : a_y \in \mathbf{R},\ y \in Y \right\}.$$

With this notation in place, it is not hard to see using (2.4) that the aforesaid condition (b) is in fact equivalent to the following:

(b′) *Let Y be any element in $\mathscr{B}(\Xi)$. For $\xi \in (\Xi \backslash Y)$,*

$$\xi = \sum_{y \in Y} a_y y \Rightarrow \lambda_\xi - \sum_{y \in Y} a_y \lambda_y \notin 2\pi i \mathbf{Z} \backslash \{0\}.$$

3. On a Property of Matrices

We commence this section by recalling a definition from [7].

Definition 3.1. An $s \times n$ *integer matrix* is called *weakly unimodular* if for each $s \times k$ $(k \leqslant s-1)$ submatrix W of Ξ with full rank (i.e., the columns of W are linearly independent), the greatest common divisor (g.c.d.) of all its $k \times k$ minors is 1.

The following proposition is stated in [7, Remark 1.1]. We include a proof here for the sake of completeness.

Proposition 3.2. *Let Ξ be an $s \times n$ integer matrix. Then Ξ is weakly unimodular if and only if every $s \times k$ $(k \leqslant s-1)$ submatrix of Ξ with full rank can be completed to an $s \times s$ unimodular integer matrix.*

Proof. To show sufficiency, let W be any $s \times k$ $(k \leqslant s-1)$ submatrix of Ξ with full rank. Then there exists an $s \times (s-k)$ integer matrix W_1 such that the completed matrix $\tilde{W} = [WW_1]$ is unimodular. Expanding $\det \tilde{W}$ $(= \pm 1)$ with respect to the $k \times k$ minors of W, we conclude that the g.c.d. of all these minors is 1.

The reverse implication is a consequence of the more general fact (see [13, p. 38]) that any $s \times k$ $(k \leqslant s-1)$ integer matrix W with full rank can be completed to an $s \times s$ integer matrix the absolute value of whose determinant equals that of the greatest common divisor of all the $k \times k$ minors of W. ∎

Motivated by Definition 3.1 and Proposition 3.2, we introduce the following

Definition 3.3. An $s \times n$ *rational matrix* Ξ is said to have *property* $\mathscr{E}$ (extendibility) if every $s \times k$ $(k \leqslant s-1)$ submatrix W of Ξ with full rank can be completed to an invertible $s \times s$ rational matrix whose inverse has only integer entries. (Note that it actually suffices to check the extendibility criterion only for $k = \min\{s-1, \operatorname{rank} \Xi\}$.)

Remark 3.4. If V is a unimodular $s \times s$ integer matrix, then it is not hard to see that Ξ has property $\mathscr{E}$ if and only if $V\Xi$ has the same property.

We now show that property $\mathscr{E}$ is an appropriate extension of the notion of weak unimodularity to rational matrices.

Proposition 3.5. *Let Ξ be an $s \times n$ integer matrix. Then Ξ has property $\mathscr{E}$ if and only if it is weakly unimodular.*

Proof. If Ξ is weakly unimodular, then it has property $\mathscr{E}$ by Proposition 3.2 (since the inverse of any unimodular integer matrix is again an integer matrix). As to the converse, suppose that Ξ has property $\mathscr{E}$ and that W is an $s \times k$ $(k \leqslant s-1)$ submatrix of Ξ with full rank. Then there exists an $s \times (s-k)$ rational matrix W_1 such that the inverse of the completed matrix $\tilde{W} = [WW_1]$ is an integer matrix. Since

$$(\tilde{W})^{-1} [WW_1] = I_s,$$

the $s \times s$ identity matrix, it follows from the Cauchy–Binet theorem (see for instance [13, p. 25]) that there exist integers λ_j (which are in fact minors of $(\tilde{W})^{-1}$) such that $\sum \lambda_j m_j = 1$, where $\{m_j\}$ is the set of all $k \times k$ minors of W. This shows that g.c.d. $(m_j) = 1$; i.e., Ξ is weakly unimodular. ∎

We observe in passing that *every* $1 \times n$ rational matrix possesses property $\mathscr{E}$ (vacuously); the following result characterizes property $\mathscr{E}$ when $s = 2$.

PROPOSITION 3.6. *Let Ξ be a $2 \times n$ rational matrix with non-zero columns $x_1, ..., x_n$. Let $x_i = 1/q_i(p_i^1, p_i^2)^{\mathrm{T}}$, $i = 1, 2, ..., n$, where $p_i^1, p_i^2, q_i \in \mathbf{Z}$. Then Ξ has property $\mathscr{E}$ if and only if for each i, $1 \leqslant i \leqslant n$, g.c.d. (p_i^1, p_i^2) divides q_i.*

Proof. Suppose Ξ has property $\mathscr{E}$. Then each column x_i can be completed to a 2×2 rational matrix whose inverse is an integer matrix. Consequently, there exist integers a_i^1, a_i^2 such that $a_i^1 p_i^1 + a_i^2 p_i^2 = q_i$; ergo, g.c.d. (p_i^1, p_i^2) divides q_i.

On the other hand, suppose that g.c.d.(p_i^1, p_i^2) divides q_i and choose $b_i^1, b_i^2 \in \mathbf{Z}$ such that $p_i^1 b_i^2 - p_i^2 b_i^1 = \text{g.c.d.}(p_i^1, p_i^2)$. Then the matrix

$$\frac{1}{q_i}\begin{pmatrix} p_i^1 & b_i^1 \\ p_i^2 & b_i^2 \end{pmatrix}$$

is a completion of x_i and its inverse has only integer entries. This being true for every i, $1 \leqslant i \leqslant n$, we conclude that Ξ has property $\mathscr{E}$. ∎

In general, if W is an $s \times k$ $(k \leqslant s-1)$ rational matrix of full rank, then $W = (1/q(W))\, W'$, where $q(W) \in \mathbf{Z} \backslash \{0\}$ and W' is an $s \times k$ integer matrix of full rank. As noted earlier, W' can be completed to an $s \times s$ integer matrix $\tilde{W}'$ such that $|\det \tilde{W}'| = d_k(W') := \text{g.c.d.}$ ($k \times k$ minors of W'). So an $s \times n$ rational matrix Ξ has property $\mathscr{E}$ if for each $s \times k$ $(k \leqslant s-1)$ submatrix W of Ξ with full rank, $d_k(W')$ divides $q(W)$. (If so, the matrix $\tilde{W} = (1/q(W))\, \tilde{W}'$ is a completion of W and $(\tilde{W})^{-1}$ has only integer entries.)

We end this section with a simple result which is used later.

PROPOSITION 3.7. *Let A be an $s\times n$ rational matrix with non-zero columns and of the form*

$$A=\begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_n^1 \\ 0 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_2^s & \cdots & a_n^s \end{pmatrix}.$$

Define A_1 to be the $s\times(n-1)$ submatrix of A whose columns comprise $(a_j^1, ..., a_j^s)^{\mathrm{T}}$, $j=2, 3, ..., n$, and let A_2 be the $(s-1)\times(n-1)$ submatrix of A with columns $(a_j^2, ..., a_j^s)^{\mathrm{T}}$, $j=2, 3, ..., n$. Assume that A has property $\mathscr{E}$. Then

(i) *A_1 and A_2 both have property $\mathscr{E}$;*

(ii) *$(a_1^1)^{-1}\in\mathbf{Z}$.*

Proof. (i) That A_1 has property $\mathscr{E}$ is quite evident. Turning to A_2, let W be any $(s-1)\times k$ $(k\leqslant s-2)$ submatrix of A_2 with full rank, say,

$$W=\begin{pmatrix} a_{r_1}^2 & \cdots & a_{r_k}^2 \\ \vdots & \ddots & \vdots \\ a_{r_1}^s & \cdots & a_{r_k}^s \end{pmatrix}.$$

The $s\times(k+1)$ matrix W_0 with columns $(a_1^1, 0, ..., 0)^{\mathrm{T}}$, $(a_{r_j}^1, ..., a_{r_j}^s)^{\mathrm{T}}$, $j=1, ..., k$ has full rank and since A has property $\mathscr{E}$, W_0 can be completed to an $s\times s$ rational matrix $\tilde{W}_0$ whose inverse is an integer matrix. Let

$$\tilde{W}_0=\begin{pmatrix} a_1^1 & a_{r_1}^1 & \cdots & a_{r_k}^1 & b_1^1 & \cdots & b_{s-k-1}^1 \\ 0 & a_{r_1}^2 & \cdots & a_{r_k}^2 & b_1^2 & \cdots & b_{s-k-1}^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{r_1}^s & \cdots & a_{r_k}^s & b_1^s & \cdots & b_{s-k-1}^s \end{pmatrix}.$$

Then the $(s-1)\times(s-1)$ matrix $\tilde{W}$ with columns $(a_{r_j}^2, ..., a_{r_j}^s)^{\mathrm{T}}, j=1, 2, ..., k$, $(b_j^2, ..., b_j)^{\mathrm{T}}$, $j=1, 2, ..., s-k-1$, is invertible and its inverse has only integer entries. This shows that A_2 has property $\mathscr{E}$.

(ii) There exists an $s\times s$ rational matrix Y with first column $(a_1^1, 0, ..., 0)^{\mathrm{T}}$ and whose inverse is an integer matrix. It follows that $(a_1^1)^{-1}\in\mathbf{Z}$. ∎

4. UPON THE DIMENSIONALITY OF $K(B_{\Xi,\lambda})$

Having discussed, in Section 2, conditions under which the space $K(B_{\Xi,\lambda})$ is trivial, we now take up the question of its finite dimensionality. The first result in this direction was established for polynomial box splines in [5] where a necessary and sufficient condition for the finite dimensionality of $K(B_\Xi)$ was obtained subject to the proviso that Ξ is an integer matrix with rank s. (The sufficiency part is contained in Theorem 4.2 of that paper whereas the necessity is implicit in the remarks following Example 4.1.) In essence, this theorem can be stated as follows [7]:

THEOREM 4.1. *Let Ξ be an $s \times n$ integer matrix with rank s and non-zero columns. Then $K(B_\Xi)$ is finite dimensional if and only if Ξ is weakly unimodular.*

An analogous theorem for exponential box splines, announced in [8], states that for $\lambda \in \mathbf{C}^n$ and Ξ as in Theorem 4.1, $K(B_{\Xi,\lambda})$ is finite dimensional provided Ξ is weakly unimodular. However, this result is true only if $\lambda \in \mathbf{R}^n$ but not otherwise. As an example, let

$$\Xi = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\lambda = (2\pi(1+2i), 2\pi(1+i), 0)$. Clearly Ξ is weakly unimodular (in fact Ξ is unimodular, i.e., $|\det Y| = 1$ for every $Y \in \mathscr{B}(\Xi)$) and has rank 2. Nonetheless, one can show that for any $\theta_2 \in \mathbf{C}$, $(-2\pi i, \theta_2)$ belongs to $N(B_{\Xi,\lambda})$. This shows, via Theorem 1.1, that $K(B_{\Xi,\lambda})$ is infinite dimensional. Subsequent to our communicating this to the authors of [8], we have received from them a corrected version of the said result [9]. It is included here (Theorem 4.3) upon their request and with their gracious permission.

The next result serves as prelude to what follows. For polynomial box splines, one direction of its proof is implicit in the remarks following [5, Example 4.1] whereas the other direction is contained in the proof of [5, Theorem 4.2] and also in that of [7, Theorem 1.1]. The appropriate modifications needed to carry the proof over to exponential box splines can be made with ease because the aforementioned arguments rely on certain common features shared by both polynomial and exponential box splines associated with integer matrices. Such details are therefore omitted.

The notation used in the sequel is in keeping with that introduced in Remark 2.13.

THEOREM 4.2. *Let Ξ be an $s \times n$ integer matrix with rank s and non-zero columns and suppose that $\lambda \in \mathbf{C}^n$ is given. Then $K(B_{\Xi,\lambda})$ is finite dimensional if and only if for each $s \times k$ submatrix Y of Ξ with rank $Y < s$, the translates $\{B_{Y,\lambda_Y}(\cdot - j) : j \in \mathbf{Z}^s\}$ are linearly independent.*

The following theorem is proved in [9]. (Compare condition (b) below with condition (b′) set out in Remark 2.13.)

THEOREM 4.3. *Let Ξ be an $s \times n$ integer matrix with non-zero columns $x_1, ..., x_n$ and rank s. Suppose that $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbf{C}^n$ is given. Then $K(B_{\Xi,\lambda})$ is finite dimensional if and only if the following conditions hold:*

(a) *Ξ is weakly unimodular;*

(b) *Let Y be any $s \times (s-1)$ submatrix of Ξ with full rank (i.e., the columns of Y are linearly independent). For $\xi \in (\Xi \backslash Y) \cap \langle Y \rangle$,*

$$\xi = \sum_{y \in Y} a_y y \Rightarrow \lambda_\xi - \sum_{y \in Y} a_y \lambda_y \notin 2\pi i \mathbf{Z} \backslash \{0\}.$$

Proof. The necessity of weak unimodularity is proved in [8, Theorem 3.1]. The necessity of condition (b) is proved in [9] as follows.

Let Y be any $s \times (s-1)$ submatrix of Ξ with linearly independent columns $x_{r_1}, ..., x_{r_{s-1}}$. Choose $\theta \in \mathbf{C}^s$ such that

$$ix_{r_k} \cdot \theta - \lambda_{r_k} = 0, \qquad k = 1, ..., s-1. \tag{4.1}$$

By Theorem 4.2, the translates $\{B_{Y,\lambda_Y}(\cdot - j) : j \in \mathbf{Z}^s\}$ are linearly independent and so (2.5) warrants the existence of an $l \in \mathbf{Z}^s$ such that

$$ix_j \cdot (\theta + 2\pi l) - \lambda_j \notin 2\pi i \mathbf{Z} \backslash \{0\}, \qquad j = 1, ..., n. \tag{4.2}$$

Since $ix_{r_k} \cdot 2\pi l \in 2\pi i \mathbf{Z}$, (4.1) and (4.2) imply that

$$x_{r_k} \cdot l = 0, \qquad k = 1, ..., s-1. \tag{4.3}$$

Now, let $\xi \in (\Xi \backslash Y) \cap \langle Y \rangle$ with

$$\xi = \sum_{k=1}^{s-1} a_k x_{r_k}. \tag{4.4}$$

Then, using (4.1), (4.3), and (4.4), we have

$$\begin{aligned} \lambda_\xi - \sum_{k=1}^{s-1} a_k \lambda_{r_k} &= \lambda_\xi - \sum_{k=1}^{s-1} i a_k x_{r_k} \cdot (\theta + 2\pi l) \\ &= \lambda_\xi - i\xi \cdot (\theta + 2\pi l). \end{aligned} \tag{4.5}$$

From (4.5) and (4.2), we conclude that

$$\lambda_\xi - \sum_{k=1}^{s-1} a_k \lambda_{r_k} \notin 2\pi i \mathbf{Z} \backslash \{0\}$$

as desired.

The proof of the sufficiency part, as detailed in [9], relies on the sufficiency part of Theorem 4.2; we do not reproduce this argument here.

Instead, we deduce the desired result from Proposition 3.5 and the forthcoming Theorem 4.7. The proof of the latter does not involve Theorem 4.2.

Remark 4.4. If Ξ is any $s \times n$ matrix and $\lambda \in \mathbf{C}^n$, then $B_{\Xi,\lambda}$ satisfies condition (b) of Theorem 4.3 if and only if $B_{V\Xi,\lambda}$ satisfies the same condition for any $s \times s$ invertible matrix V.

Also in connection with condition (b), the following observation is made in [9]. Suppose that Ξ is an $s \times n$ real matrix and Y an $s \times k$ submatrix of Ξ with linearly independent columns, x_{r_j}, $j = 1, ..., k$. Assume that

$$\xi = (\xi^1, ..., \xi^s) \in (\Xi \backslash Y) \cap \langle Y \rangle$$

with

$$\xi = \sum_{j=1}^{k} a_j x_{r_j}.$$

Then there exists a non-singular $k \times k$ submatrix Y' of Y with columns say, $(x_{r_j}^{l_1}, ..., x_{r_j}^{l_k})^{\mathrm{T}}$, $j = 1, ..., k$, such that

$$\sum_{j=1}^{k} a_j x_{r_j}^{l_p} = \xi^{l_p}, \qquad p = 1, ..., k.$$

Defining

$$\tilde{Y}' := \begin{pmatrix} x_{r_1}^{l_1} & \cdots & x_{r_k}^{l_1} & \xi^{l_1} \\ \vdots & \ddots & \vdots & \vdots \\ x_{r_1}^{l_k} & \cdots & x_{r_k}^{l_k} & \xi^{l_k} \\ \lambda_{r_1} & \cdots & \lambda_{r_k} & \lambda_{\xi} \end{pmatrix},$$

it is not hard to see that

$$\lambda_{\xi} - \sum_{j=1}^{k} a_j \lambda_{r_j} = \frac{\det \tilde{Y}'}{\det Y'}.$$

This provides a readily computable method for checking condition (b).

Turning now to rational matrices, we begin with the one-dimensional case. The following theorem is a special case of [15, Corollary 2.4(a)] but we provide a self-contained proof for the sake of completeness.

THEOREM 4.5. *Let Ξ be a $1 \times n$ rational matrix with non-zero columns $x_1, ..., x_n$ and let $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbf{C}^n$. Then $K(B_{\Xi,\lambda})$ is always finite dimensional.*

Proof. In view of Theorem 1.1, it suffices to show that the set

$$S := \{\theta = 2\pi(\alpha + i\beta) \in N(B_{\Xi,\lambda}) : \alpha, \beta \in \mathbf{R}, 0 \leqslant \alpha < 1\}$$

is finite. We let $x_k = p_k/q_k$, $k = 1, ..., n$, where $p_k, q_k \in \mathbf{Z}$ and $p_k > 0$. Let us suppose that $\lambda_k = 2\pi(\sigma_k + it_k)$, $k = 1, ..., n$, $\sigma_k, t_k \in \mathbf{R}$, and define

$$R_k := \left\{\frac{j + t_k q_k}{p_k} : j \in \mathbf{Z}, \ -t_k q_k \leqslant j < p_k - t_k q_k\right\}.$$

Further, let

$$\mathfrak{R} := \bigcup_{k=1}^{n} R_k$$

and

$$\mathfrak{I} := \left\{\frac{-\sigma_k q_k}{p_k} : 1 \leqslant k \leqslant n\right\}.$$

Now let $\theta = 2\pi(\alpha + i\beta) \in S$. By (2.2), there exists a k, $1 \leqslant k \leqslant n$, such that

$$i\left(\sigma_k + \frac{\beta p_k}{q_k}\right) + \left(\alpha - \frac{t_k q_k}{p_k}\right)\frac{p_k}{q_k} \in \mathbf{Z}\backslash\{0\}.$$

It follows that

$$\sigma_k + \frac{\beta p_k}{q_k} = 0 \tag{4.6}$$

and

$$\alpha p_k - t_k q_k \in \mathbf{Z}. \tag{4.7}$$

From (4.6), (4.7), and the fact that $0 \leqslant \alpha < 1$, we conclude that

$$\alpha \in \mathfrak{R} \qquad \text{and} \qquad \beta \in \mathfrak{I}.$$

This finishes the proof because both $\mathfrak{R}$ and $\mathfrak{I}$ are finite sets. ∎

Remark 4.6. If V is a unimodular $s \times s$ integer matrix, then so is V^{-1}. Therefore, by (2.11), the sequence $(a(j))_{j \in \mathbf{Z}^s} \in K(B_{\Xi,\lambda})$ if and only if $(a(V^{-1}j))_{j \in \mathbf{Z}^s} \in K(B_{V\Xi,\lambda})$. Consequently, either one of these spaces is finite dimensional precisely when the other is.

We now pass to higher dimensions. The following is an improved version of our earlier result in this direction, thanks to condition (b) of Theorem 4.3 whose use was suggested by [9]. In what follows, if $\phi = (\phi_1, \phi_2, ..., \phi_s) \in \mathbf{C}^s$, then $P(\phi)$ will denote its projection $(\phi_2, ..., \phi_s) \in \mathbf{C}^{s-1}$.

THEOREM 4.7. *Let Ξ be an $s \times n$ rational matrix with non-zero columns $x_1, ..., x_n$ and rank s. Let $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbf{C}^n$ be given and suppose that the following two conditions hold*:

(a) *Ξ has property $\mathscr{E}$*;

(b) *Let Y be any $s \times (s-1)$ submatrix of Ξ with full rank. For $\xi \in (\Xi \backslash Y) \cap \langle Y \rangle$,*

$$\xi = \sum_{y \in Y} a_y y \Rightarrow \lambda_\xi - \sum_{y \in Y} a_y \lambda_y \notin 2\pi i \mathbf{Z} \backslash \{0\}.$$

Then $K(B_{\Xi,\lambda})$ is finite dimensional.

Proof. We use induction on s and n. Theorem 4.5 covers the case $s=1$. We may therefore assume the validity of the result for $s-1$.

Now for the case $n=s$. It may be assumed that Ξ is of the form

$$\Xi = \begin{pmatrix} x_1^1 & x_2^1 & \cdots & x_s^1 \\ 0 & x_2^2 & \cdots & x_s^2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & x_2^s & \cdots & x_s^s \end{pmatrix}.$$

(Otherwise, we can find a unimodular $s \times s$ integer matrix V such that $V\Xi$ has the form above. This matrix also satisfies conditions (a) (Remark 3.4) and (b) (Remark 4.4) and it suffices to prove that $K(B_{V\Xi,\lambda})$ is finite dimensional (Remark 4.6).)

Note that Proposition 3.7(ii) guarantees that

$$(x_1^1)^{-1} \in \mathbf{Z}. \tag{4.8}$$

We need to show (by Theorem 1.1) that $N(B_{\Xi,\lambda})/2\pi\mathbf{Z}^s$ is a finite set. Let $\theta = (\theta_1, ..., \theta_s) \in N(B_{\Xi,\lambda})$, $\theta_j = 2\pi(\alpha_j + i\beta_j)$, $\alpha_j, \beta_j \in \mathbf{R}$, and $0 \leqslant \alpha_j < 1$. Also suppose that $\lambda_k = 2\pi(\sigma_k + it_k)$, $\sigma_k, t_k \in \mathbf{R}$, $k = 1, ..., s$.

Let Ξ' be the $s \times (s-1)$ matrix with columns x_k, $k = 2, ..., s$, and $\lambda' := P(\lambda)$. Since Ξ has property $\mathscr{E}$, there exists $z = (z^1, ..., z^s) \in \mathbf{R}^s$ such that the $s \times s$ matrix $\tilde{\Xi}'$ with columns x_k, $k = 2, ..., s, z$ (in that order), has an inverse all of whose entries are integers. Defining $\tilde{\lambda}' := (\lambda_2, ..., \lambda_s, 0)$, it follows from Remark 2.3(ii) and Theorem 2.9 that $N(B_{\tilde{\Xi}',\tilde{\lambda}'})$ is empty. As $N(B_{\Xi',\lambda'}) \subseteq N(B_{\tilde{\Xi}',\tilde{\lambda}'})$ and $\theta \in N(B_{\Xi,\lambda})$, there exists a $d = (d_1, ..., d_s) \in \mathbf{Z}^s$ such that

$$i\frac{\lambda_1}{2\pi} + \left(\frac{\theta}{2\pi} + d\right) \cdot x_1 \in \mathbf{Z} \backslash \{0\}. \tag{4.9}$$

Therefore,

$$\sigma_1 + \beta_1 x_1^1 = 0 \tag{4.10}$$

and since $(x_1^1)^{-1}$ and d_1 are integers,

$$\alpha_1 - \frac{t_1}{x_1^1} \in \mathbf{Z}. \tag{4.11}$$

Conditions (4.10), (4.11), and the fact that $0 \leqslant \alpha_1 < 1$ fix θ_1 uniquely. Also note that (4.8) and (4.9) imply

$$i\frac{\lambda_1}{2\pi x_1^1} + \frac{\theta_1}{2\pi} \in \mathbf{Z}. \tag{4.12}$$

Now, let Ξ'' be the $(s-1)\times(s-1)$ submatrix of Ξ with non-zero columns $P(x_k)$, $k=2, ..., s$, and rank $s-1$. Define $\mu := (\mu_2, ..., \mu_s) \in \mathbf{C}^{s-1}$, where

$$\mu_k := \lambda_k - \frac{\lambda_1}{x_1^1} x_k^1, \qquad k=2, ..., s. \tag{4.13}$$

The matrix Ξ'' has property $\mathscr{E}$ by Proposition 3.7(i) and $B_{\Xi'',\mu}$ satisfies condition (b) vacuously because Ξ'' has rank $s-1$. Let $(l_2, ..., l_s) \in \mathbf{Z}^{s-1}$ and set

$$l_1 := -i\frac{\lambda_1}{2\pi x_1^1} - \frac{\theta_1}{2\pi}.$$

For $l = (l_1, l_2, ..., l_s) \in \mathbf{Z}^s$, there is a k (by (2.2)) such that $1 \leqslant k \leqslant s$ and

$$i\frac{\lambda_k}{2\pi} + \left(\frac{\theta}{2\pi} + l\right)\cdot x_k \in \mathbf{Z}\backslash\{0\}. \tag{4.14}$$

By choice of l_1, $2 \leqslant k \leqslant s$; thus, from (4.14), (4.13), and the fact that $l_2, ..., l_s$ are arbitrary, we conclude that

$$P(\theta) \in N(B_{\Xi'',\mu})/2\pi\mathbf{Z}^{s-1}$$

which set, by the induction hypothesis, is finite.

This completes the proof for the case $n=s$. Assume now that the theorem is true for $n-1$.

Although the proof of the general case is quite similar to that of the previous one, there are some notable differences that ought to be dealt with in some detail.

To proceed, we may assume once again that Ξ is of the form

$$\Xi = \begin{pmatrix} x_1^1 & x_2^1 & \cdots & x_n^1 \\ 0 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & x_2^s & \cdots & x_n^s \end{pmatrix}$$

with $(x_1^1)^{-1} \in \mathbf{Z}$. We may also assume that the $s \times (n-1)$ matrix with columns x_k, $k = 2, ..., n$, has rank s. By rearranging, if need be, columns 2 through n (*and* the corresponding λ_k's) we may assume that for each $k = 2, ..., m$, $P(x_k) \neq 0$ whereas for $k = m+1, ..., n$, $P(x_k) = 0$. We designate Ξ' to be the $s \times (n-1)$ submatrix of Ξ with columns x_k, $k = 2, ..., n$, and $\lambda' = (\lambda_2, ..., \lambda_n)$. Note that Ξ' has property $\mathscr{E}$ (Proposition 3.7(ii)) and $B_{\Xi', \lambda'}$ satisfies condition (b) because $B_{\Xi, \lambda}$ does. By the induction hypothesis applied to $B_{\Xi', \lambda'}$, it suffices to show that the set $(N(B_{\Xi, \lambda}) \backslash N(B_{\Xi', \lambda'}))/2\pi \mathbf{Z}^s$ is finite. If $\theta = (\theta_1, ..., \theta_s)$ belongs to this set, then there must be a $d = (d_1, ..., d_s) \in \mathbf{Z}^s$ for which (4.9) holds. Once again this fixes θ_1 uniquely and (4.12) holds as well. Choosing integers $l_2, ..., l_s$ arbitrarily and l_1 as before, we conclude that (4.14) holds for some k, $2 \leqslant k \leqslant n$. Suppose now that $k \in \{m+1, ..., n\}$. We can find an $s \times (s-1)$ submatrix Y of Ξ with full rank and whose first column is x_1. Clearly,

$$x_k \in (\Xi \backslash Y) \cap \langle Y \rangle \qquad \text{and} \qquad x_k = \frac{x_k^1}{x_1^1} x_1. \tag{4.15}$$

Further,

$$i \frac{\lambda_k}{2\pi} + \left(\frac{\theta}{2\pi} + l \right) \cdot x_k = i \frac{\lambda_k}{2\pi} - i \frac{\lambda_1}{2\pi x_1^1} x_k^1. \tag{4.16}$$

From (4.15), condition (b), and (4.16), we conclude that (4.14) does not hold for any $k \in \{m+1, ..., n\}$. It must therefore hold for some $k \in \{2, ..., m\}$.

We now pass to the $(s-1) \times (m-1)$ matrix Ξ'' whose columns comprise $P(x_k)$, $k = 2, ..., m$, and define $\mu := (\mu_2, ..., \mu_m) \in \mathbf{C}^{m-1}$, where μ_k is given by (4.13) for each $k = 2, ..., m$. The preceding arguments have in effect shown that

$$P(\theta) \in N(B_{\Xi'', \mu})/2\pi \mathbf{Z}^{s-1}.$$

In order to apply the induction hypothesis to $B_{\Xi'', \mu}$ and thereby complete the proof, it remains to demonstrate that $B_{\Xi'', \mu}$ satisfies the conditions of the theorem.

To that end, we first note that Ξ'' has non-zero columns and rank $s-1$ and it satisfies condition (a) by Proposition 3.7(i). As for condition (b), let Y'' be an $(s-1) \times (s-2)$ submatrix of Ξ'' with linearly independent columns $P(x_{r_k})$, $k = 2, ..., s-1$. Assume that $P(\xi) \in (\Xi'' \backslash Y'') \cap \langle Y'' \rangle$, $\xi = (\xi^1, ..., \xi^s) \in \Xi$, and

$$P(\xi) = \sum_{k=2}^{s-1} a_k P(x_{r_k}). \tag{4.17}$$

Define $r_1 := 1$ and find $a_1 \in \mathbf{R}$ such that

$$\sum_{k=1}^{s-1} a_k x_{r_k}^1 = \xi^1. \tag{4.18}$$

The $s \times (s-1)$ submatrix Y of Ξ with columns x_{r_k}, $k = 1, ..., s-1$, is of full rank and it is clear that $\xi \in (\Xi \backslash Y)$. Further, it follows from (4.17) and (4.18) that

$$\xi = \sum_{k=1}^{s-1} a_k x_{r_k}. \tag{4.19}$$

Since $B_{\Xi,\lambda}$ satisfies condition (b), (4.19) implies that

$$\lambda_\xi - \sum_{k=1}^{s-1} a_k \lambda_{r_k} \notin 2\pi i \mathbf{Z} \backslash \{0\}. \tag{4.20}$$

From (4.13) and (4.18) it is easy to see that

$$\mu_{P(\xi)} - \sum_{k=2}^{s-1} a_k \mu_{r_k} = \lambda_\xi - \sum_{k=1}^{s-1} a_k \lambda_{r_k}. \tag{4.21}$$

Thus (4.21) and (4.20) show that $B_{\Xi'',\mu}$ satisfies condition (b) and this completes the proof. ∎

As promised earlier, the preceding theorem, taken in conjunction with Proposition 3.5, completes the sufficiency part of Theorem 4.3. We also note that condition (b) of Theorem 4.7 is met automatically if $\lambda \in \mathbf{R}^n$. This leads us to

COROLLARY 4.8. *Let Ξ be an $s \times n$ rational matrix with non-zero columns and rank s. Suppose that $\lambda \in \mathbf{R}^n$ is given. Then $K(B_{\Xi,\lambda})$ has finite dimension provided Ξ has property $\mathscr{E}$.*

Corollary 4.8 and Proposition 3.5 clearly serve to extend the sufficiency part of Theorem 4.1 to rational matrices.

Remark 4.9. Condition (a) of Theorem 4.7 is not necessary for finite dimensionality unless Ξ is an integer matrix. Take, for example, Ξ to be the matrix given in Remark 2.11 and let $\lambda = 0$. Then $K(B_{\Xi,\lambda})$ is trivial, *a fortiori*, finite dimensional. However, Proposition 3.6 shows that Ξ does not have property $\mathscr{E}$. Of course, if Ξ is an integer matrix, then the necessity of property $\mathscr{E}$ follows from Theorem 4.3 and Proposition 3.5.

5. An Application

Throughout this final section, we assume that Ξ is an $s \times n$ integer matrix and $\lambda = 0$. If $\mathscr{A}$ is a (perhaps proper) submodule of the **Z**-module $\mathbf{Z}^s$, then we define the sequence space

$$K_{\mathscr{A}}(B_\Xi) := \left\{ a : \mathscr{A} \to \mathbf{C} : \sum_{k \in \mathscr{A}} a(k)\, B_\Xi(\cdot - k) = 0 \right\}.$$

If $\mathscr{A}$ is a proper submodule of $\mathbf{Z}^s$, then it is clear from the definition above that any sequence in $K_{\mathscr{A}}(B_\Xi)$ can be canonically extended to one in $K(B_\Xi)$ simply by assigning the value zero at those lattice points in $\mathbf{Z}^s \backslash \mathscr{A}$. So if $K(B_\Xi)$ is finite dimensional, then so too is $K_{\mathscr{A}}(B_\Xi)$. Nonetheless, it is quite conceivable that for a properly chosen submodule $\mathscr{A}$, the space $K_{\mathscr{A}}(B_\Xi)$ has finite dimension while $K(B_\Xi)$ itself may be infinite dimensional. This next result helps identify such submodules. (Compare [11, Corollary 2.6].)

Proposition 5.1. *Let Ξ be an $s \times n$ integer matrix with rank s and non-zero columns. Let $\mathscr{A} := A\mathbf{Z}^s$, where A is an invertible $s \times s$ integer matrix. Assume that each $s \times k$ $(k \leqslant s-1)$ submatrix W of Ξ with full rank can be completed to an $s \times s$ integer matrix $\tilde{W}$ such that $\mathscr{A} \subseteq \tilde{W}\mathbf{Z}^s$. Then $K_{\mathscr{A}}(B_\Xi)$ is finite dimensional.*

Proof. Defining $\Xi' := A^{-1}\Xi$, it is not hard to see, using (2.11), that $K_{\mathscr{A}}(B_\Xi)$ is finite dimensional if and only if $K(B_{\Xi'})$ is. Evidently Ξ' is a rational matrix and if W' is any $s \times k$ submatrix of Ξ' with full rank, then $W' = A^{-1}W$ for some $s \times k$ submatrix W of Ξ with full rank. By our premise, W can be completed to an $s \times s$ integer matrix $\tilde{W}$ such that $A\mathbf{Z}^s \subseteq \tilde{W}\mathbf{Z}^s$; i.e., there is an $s \times s$ integer matrix Q such that $A = \tilde{W}Q$. Now $\tilde{W}' = A^{-1}\tilde{W}$ is a completion of W' and $(\tilde{W}')^{-1} = Q$, an integer matrix. This shows that Ξ' has property $\mathscr{E}$ and the desired result follows from Corollary 4.8. ∎

Observe that if $\mathscr{A} = \mathbf{Z}^s$, then the assumption in the last proposition, in view of Proposition 3.2, is equivalent to weak unimodularity of Ξ and thereby to the finite dimensionality of $K_{\mathscr{A}}(B_\Xi)$ $(= K(B_\Xi))$.

Example 5.2. Let

$$\Xi := \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 4 \end{pmatrix},$$

$$A := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \mathscr{A} := A\mathbf{Z}^2.$$

As Ξ is not weakly unimodular, $K(B_\Xi)$ is infinite dimensional. However $K_{\mathscr{A}}(B_\Xi)$ has finite dimension. This can be seen from Proposition 5.1 or by noting that

$$A^{-1}\Xi =: \Xi' = \begin{pmatrix} \frac{1}{2} & 0 & 3 \\ \frac{1}{2} & 1 & -1 \end{pmatrix}$$

has property $\mathscr{E}$ (Proposition 3.6).

Also note that for the submatrix

$$Y := \begin{pmatrix} 0 & 3 \\ 1 & -1 \end{pmatrix}$$

of Ξ', $|\det Y| = 3$ and so the integer translates of B_Y are linearly dependent [3, Proposition 4 or Theorem 2.1]. Thus $N(B_Y)$ is non-empty and therefore $N(B_{\Xi'})$ ($\supseteq N(B_Y)$) is non-void as well. This shows that $K(B_{A^{-1}\Xi})$ and hence $K_{\mathscr{A}}(B_\Xi)$ is non-trivial.

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